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A generalization of Lyapounov's Theorem

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ABSTRACT

Let $\vec{\mu}=(\mu_1,\dots,\mu_n)$ be a vector measure on a measurable space $(\Omega,\mathscr{F})$ such that each μ_i is finite. The measures are allowed to have atoms. In this paper conditions on $\vec{\mu}$ are given under which the matrix-k-range $\mathscr{MR}_k(\vec{\mu})=\{((\mu_i(P_j))_{i=1}^n)_{j=1}^k:\{P_j\}_{j=1}^k$ is a measurable partition of $\Omega\}$ is convex.

This will lead to conditions on $\vec{\mu}$ under which the partition-range $\mathscr{PR}(\vec{\mu})=\{(\mu_1(P_1),\dots,\mu_n(P_n)):\{P_i\}_{i=1}^n$ is a measurable partition of $\Omega\}$ and the range $\mathscr{R}(\vec{\mu})=\{(\mu_1(B),\dots,\mu_n(B)):B\in\mathscr{F}\}$ are convex. The results (i.e. Theorem 2.7 and Theorem 2.8) are generalizations of Lyapounov's Theorem.

1. INTRODUCTION

Let $\mu_1,\dots,\mu_n$ be countably additive measures on a measurable space $(\Omega,\mathscr{F})$. Throughout this paper we only consider nonnegative finite measures. It is also assumed that the measures are nontrivial, i.e. $\mu_j\not\equiv 0$, $j=1,\dots,n$.

DEFINITION 1.1. $\mathscr{R}(\vec{\mu}):=\{(\mu_1(B),\dots,\mu_n(B)):B\in\mathscr{F}\}$ is the range of the vector measure $\vec{\mu}=(\mu_1,\dots,\mu_n)$.

In 1940, Lyapounov [5] proved the following theorem:

THEOREM 1.2. *If $\mu_1,\dots,\mu_n$ are nonatomic measures on $(\Omega,\mathscr{F})$, then $\mathscr{R}(\vec{\mu})$ is convex and compact.*

Lyapounov's Theorem was generalized in 1951 by Dvoretzky, Wald and Wolfowitz. In order to state their theorem a few definitions are needed.

DEFINITION 1.3. $P_1, \ldots, P_k$ is a measurable k-partition of Ω if $P_j \in \mathscr{F}$ for $j = 1, \ldots, k$; $P_l \cap P_j = \emptyset$ if $l \neq j$ and $\bigcup_{j=1}^{k} P_j = \Omega$.

Let $(\mu_i(P_j))_{i=1}^{n}{}_{j=1}^{k}$ denote the $n \times k$ matrix which has its (i,j)-th entry equal to $\mu_i(P_j)$, $i = 1, \ldots, n$, $j = 1, \ldots, k$.

DEFINITION 1.4. Let $\mathscr{MR}_k(\vec{\mu}) := \{(\mu_i(P_j))_{i=1}^{n}{}_{j=1}^{k} : \{P_j\}_{j=1}^{k}$ is a measurable k-partition of $\Omega\}$ denote the matrix-k-range of $\vec{\mu}$.

THEOREM 1.5 (Dvoretzky, Wald and Wolfowitz). *If $\mu_1, \ldots, \mu_n$ are nonatomic measures on $(\Omega, \mathscr{F})$, then $\mathscr{MR}_k(\vec{\mu})$ is a convex and compact set.*

Note that the matrix-2-range $\mathscr{MR}_2(\vec{\mu})$ and the range $\mathscr{R}(\vec{\mu})$ are strongly related. In fact, it is easy to see that $\mathscr{MR}_2(\vec{\mu})$ is convex if and only if $\mathscr{R}(\vec{\mu})$ is convex. Therefore, Theorem 1.5 is indeed a generalization of Lyapounov's Theorem. From now on the range is treated as a special case of the matrix-k-range.

We consider one more type of range. This is the so-called partition-range. The partition-range is interesting in view of the applications, as will be seen later.

DEFINITION 1.6. The set $\mathscr{PR}(\vec{\mu}) := \{(\mu_1(P_1), \ldots, \mu_n(P_n)) : \{P_j\}_{j=1}^{n}$ is a measurable n-partition of $\Omega\}$ denotes the partition-range of $\vec{\mu}$.

In Theorem 1.2 and 1.5, the measures are supposed to be nonatomic. This condition is sufficient, but not necessary, as can be seen in the next example.

EXAMPLE 1.7. Let $\Omega = [0,1]$ and let $\mathscr{F}$ be the collection of Borel sets on $[0,1]$. Let λ be the Lebesgue measure on $(\Omega, \mathscr{F})$. Define μ by

$$\begin{aligned} &\mu = \lambda && \text{on } [0, \tfrac{1}{2}], \\ &\mu(\{1\}) = \tfrac{1}{2} && \\ &\mu = 0 && \text{elsewhere.} \end{aligned}$$

In this case, μ has an atom. But still $\mathscr{R}(\mu) = [0,1]$ and thus is convex.

In this paper, a generalization of Theorem 1.5 is proved, which is in turn a generalization of Lyapounov's Theorem. We will find a larger set of measures (not only nonatomic measures) for which $\mathscr{MR}_k(\vec{\mu})$ is convex. (This is the first main theorem: Theorem 2.7.) This leads to a larger set of measures for which $\mathscr{R}(\vec{\mu})$ (see Corollary 2.9) and $\mathscr{PR}(\vec{\mu})$ (see Theorem 2.8, which is the second main theorem) are convex.

The rest of this paper is organized as follows. In Section 2, some notation is developed and the main theorems are stated. Section 3 is the technical part of this paper. It contains a method of enlarging the space and extending the measures to this new enlarged space. This notion is very important in the proof of the main theorems. In Section 4, we prove one of the main theorems, considering the matrix-k-range. We also take a look at $\mathscr{R}(\vec{\mu})$. The partition-range is considered in Section 5. Section 6 finally, contains some applications.

2. PRELIMINARIES AND MAIN RESULTS

Let $\mu_1, \ldots, \mu_n$ be measures on a measurable space $(\Omega, \mathscr{F})$. The measures are allowed to have atoms.

DEFINITION 2.1. Let μ be a measure on a measurable space $(\Omega, \mathscr{F})$. A set $E \in \mathscr{F}$ is called an atom of μ, if $\mu(E) > 0$ and for every $F \in \mathscr{F}$ with $F \subset E$ either $\mu(F) = 0$ or $\mu(F) = \mu(E)$.

A finite measure has at most countably many atoms (cf. Rényi [7]). Let A_i denote the set of atoms of μ_i ($i = 1, \ldots, n$). In this paper, we consider the special case where the measures have no atoms in common, i.e. $A_i \cap A_j = \emptyset$ if $j \neq i$.

We want to identify each atom with a single point. It will be proved that this is indeed possible. For, suppose $E \in \mathscr{F}$ is an atom of μ_i. If we want to treat E as a single point, then we need to know that $\mu_j(E) = 0$ if $j \neq i$. This can always be obtained by the following procedure. (The argument is only given for two measures, if $n > 2$ the argument is essentially the same.)

Suppose $E \in \mathscr{F}$ is an atom of μ_1 and $F \in \mathscr{F}$ is an atom of μ_2. Then we can assume that $E \cap F = \emptyset$. For, suppose $E \cap F \neq \emptyset$. Since μ_1 and μ_2 have no common atoms, either $\mu_1(E \cap F) = 0$ or $\mu_2(E \cap F) = 0$. So either E can be replaced by $E \setminus F$ or F can be replaced by $F \setminus E$.

Now consider E. If $\mu_2(E) = 0$, then E can be replaced by a single point, and there is nothing to prove. Suppose $\mu_2(E) > 0$. If we restrict μ_2 to E, then $\mu_2|_E$ is nonatomic, so by Lyapounov's Theorem there exists a set $B_1 \subset E$, such that

$$\mu_2(B_1) = \mu_2(B_1^c) = \tfrac{1}{2}\mu_2(E).$$

Also, since E is an atom of μ_1, either $\mu_1(B_1) = \mu_1(E)$ or $\mu_1(B_1^c) = \mu_1(E)$. Assume that $\mu_1(B_1) = \mu_1(E)$. Proceed by induction. It is possible to find a set $B_n \subset B_{n-1}$ such that

$$\mu_2(B_n) = \tfrac{1}{2}\mu_2(B_{n-1}) = (\tfrac{1}{2})^n \mu_2(E)$$

$$\mu_1(B_n) = \mu_1(E).$$

The sequence $\{B_n\}_{n=1}^{\infty}$ is decreasing. Consider $\bigcap_{n=1}^{\infty} B_n$. Then

$$\mu_2\Big(\bigcap_{n=1}^{\infty} B_n\Big) = \lim_{n \to \infty} \mu_2(B_n) = 0$$

$$\mu_1\Big(\bigcap_{n=1}^{\infty} B_n\Big) = \lim_{n \to \infty} \mu_1(B_n) = \mu_1(E).$$

So we can replace E by $\bigcap_{n=1}^{\infty} B_n$ to establish that the atoms of μ_1 have μ_2-measure 0.

Now we return to the convexity of the matrix-k-range. First the case where there is only one measure is considered, i.e. $n=1$.

Let μ be a measure on a measurable space $(\Omega, \mathcal{F})$. Let $A=\{x_1, x_2, \ldots\}$ denote the set of atoms of μ, with $\mu(\{x_m\}) \geq \mu(\{x_{m+1}\})$ for $m \in \mathbb{N}$. Note that A can be finite, or even empty.

If μ is nonatomic, then $\mathcal{R}(\mu)$ (and hence $\mathcal{MR}_2(\mu)$) is convex by Lyapounov's Theorem. But we also saw in Example 1.7 that this is not a necessary condition. It is maybe even more surprising that a purely atomic measure can have a convex range, as can be seen in the next example.

EXAMPLE 2.2. Let $(\Omega, \mathcal{F})=(\mathbb{R}, \textit{Borel sets})$ and let $\mu(\{n\})=(\frac{1}{2})^n$ for $n \in \mathbb{N}$. Then μ is a purely atomic probability measure. We will show that $\mathcal{R}(\mu)=[0,1]$.

Let $\beta \in [0,1]$ be given. Then β has a binary expansion: $\beta=\sum_{n=1}^{\infty} \beta_n(\frac{1}{2})^n$ with $\beta_n=0$ or 1 for all n. Let $B:=\{n \in \mathbb{N};\, \beta_n=1\}$. Then $\mu(B)=\beta$, and thus $\beta \in \mathcal{R}(\mu)$. So $\mathcal{R}(\mu)=[0,1]$.

In the one-dimensional case, it is possible to find necessary and sufficient conditions on μ such that $\mathcal{MR}_k(\mu)$ is convex. The same problem can be found in Rényi [7, p. 80–81] as exercises. We state the result here.

THEOREM 2.3.

$$\begin{aligned}&\mathcal{MR}_k(\mu) \textit{ is convex}\\ &\Leftrightarrow\ (k-1)\mu(\{x_m\}) \leq \mu(\Omega \backslash A)+\sum_{l=m+1}^{\infty} \mu(\{x_l\}) \quad \forall m \in \mathbb{N}\end{aligned}$$

and if μ is purely atomic, then A is not a finite set.

Notice that $(k-1)\mu(\{x_m\}) \leq \mu(\Omega \backslash A)+\sum_{l=m+1}^{\infty} \mu(\{x_l\})$ is equivalent with $\mu(\{x_m\}) \leq 1/k(\mu(\Omega \backslash A)+\sum_{l=m}^{\infty} \mu(\{x_l\}))$. The latter is used in Rényi [7].

Now we consider the case where $n \geq 2$. Recall that A_i was the set of atoms of μ_i. Denote $A_i:=\{x_1^{(i)}, x_2^{(i)}, \ldots\}$, where $\mu_i(\{x_m^{(i)}\}) \geq \mu_i(\{x_{m+1}^{(i)}\})$ for $m \in \mathbb{N}$ and $i=1, \ldots, n$. Note that A_i can be finite or even empty.

In order to formulate the main theorem, we need some more notation. First we need the Lebesgue decomposition of a measure, cf. [2, p. 134].

A measure μ is concentrated on a set $E \in \mathcal{F}$ if $\mu(F)=\mu(E \cap F)$ for every $F \in \mathcal{F}$. Let μ and ν be measures. Then μ and ν are singular (notation $\mu \perp \nu$) if μ and ν are concentrated on disjoint sets. We say that μ is absolutely continuous with respect to ν (notation $\mu \ll \nu$) if $\nu(F)=0$ implies that $\mu(F)=0$. We now state a decomposition theorem.

THEOREM 2.4. *Let μ and ν be measures on a measurable space $(\Omega, \mathcal{F})$. Then there exist two uniquely determined measures σ_a and σ_s on $(\Omega, \mathcal{F})$ such that $\mu=\sigma_a+\sigma_s$, where $\sigma_a \ll \nu$ and $\sigma_s \perp \nu$. Moreover $\sigma_a \perp \sigma_s$.*

Theorem 2.4 can be generalized as follows. Suppose μ_1, μ_2 and μ_3 are measures on a measurable space $(\Omega, \mathscr{F})$. Then Theorem 2.4 implies that $\mu_1 = \sigma_{oa} + \sigma_{os}$, where $\sigma_{oa} \ll \mu_2$ and $\sigma_{os} \perp \mu_2$. Now σ_{oa} and σ_{os} are also measures on $(\Omega, \mathscr{F})$. So we can make a decomposition of σ_{oa} and σ_{os} with respect to μ_3, i.e.,

$$\sigma_{oa} = \sigma_{oaa} + \sigma_{oas}, \quad \text{where } \sigma_{oaa} \ll \mu_3 \text{ and } \sigma_{oas} \perp \mu_3$$

and

$$\sigma_{os} = \sigma_{osa} + \sigma_{oss}, \quad \text{where } \sigma_{osa} \ll \mu_3 \text{ and } \sigma_{oss} \perp \mu_3.$$

Combining this, we get $\mu_1 = \sigma_{oaa} + \sigma_{oas} + \sigma_{osa} + \sigma_{oss}$, where

$$\begin{aligned} &\sigma_{oaa} \ll \mu_2 \quad \text{and} \quad \sigma_{oaa} \ll \mu_3 \\ &\sigma_{oas} \ll \mu_2 \quad \text{and} \quad \sigma_{oas} \perp \mu_3 \\ &\sigma_{osa} \perp \mu_2 \quad \text{and} \quad \sigma_{osa} \ll \mu_3 \\ &\sigma_{oss} \perp \mu_2 \quad \text{and} \quad \sigma_{oss} \perp \mu_3. \end{aligned}$$

It turns out that the decomposition will remain the same if we change the roles of μ_2 and μ_3. The next theorem extends this idea for n measures. (A proof can be found in the appendix. Theorem 2.5 and Lemma 2.6 may already be known, but no reference is known to the author.) Note that the o is just a dummy index.

THEOREM 2.5. *Let $\mu_1, \ldots, \mu_n$ be measures. Let I_i be the following index set:*

$$I_i := \{(z_1, \ldots, z_n) \mid z_j \in \{a, s\} \text{ if } j \neq i \text{ and } z_i = o\}.$$

Then μ_i can be decomposed as follows:

$$\mu_i = \sum_{(z_1, \ldots, z_n) \in I_i} \sigma^{(i)}_{(z_1, \ldots, z_n)}$$

where for $j \neq i$

$$\begin{aligned} &\sigma^{(i)}_{(z_1, \ldots, z_n)} \ll \mu_j \quad \text{if } z_j = a \\ &\sigma^{(i)}_{(z_1, \ldots, z_n)} \perp \mu_j \quad \text{if } z_j = s. \end{aligned}$$

A decomposition of μ_i with these properties is unique. Moreover, if $(z_1, \ldots, z_n)$ and $(z'_1, \ldots, z'_n) \in I_i$, and $(z_1, \ldots, z_n) \neq (z'_1, \ldots, z'_n)$, then $\sigma^{(i)}_{(z_1, \ldots, z_n)} \perp \sigma^{(i)}_{(z'_1, \ldots, z'_n)}$.

Define $S^{(i)}_{(z_1, \ldots, z_n)} := \operatorname{supp}(\sigma^{(i)}_{(z_1, \ldots, z_n)})$. (Here supp denotes the support of the measure $\sigma^{(i)}_{(z_1, \ldots, z_n)}$. The support of a measure is uniquely defined modulo a set of measure 0. We will make a choice later on.) For notational convenience, we use $S_i := S^{(i)}_{(\tilde{z}_1, \ldots, \tilde{z}_n)}$, where $\tilde{z}_j = s$ for $j \neq i$. Since for fixed i we have $\sigma^{(i)}_{(z_1, \ldots, z_n)} \perp \sigma^{(i)}_{(z'_1, \ldots, z'_n)}$ if $(z_1, \ldots, z_n) \neq (z'_1, \ldots, z'_n)$, it may be assumed that $S^{(i)}_{(z_1, \ldots, z_n)} \cap S^{(i)}_{(z'_1, \ldots, z'_n)} = \emptyset$ if $(z_1, \ldots, z_n) \neq (z'_1, \ldots, z'_n)$.

We now make some more remarks on the sets $S^{(i)}_{(z_1, \ldots, z_n)}$. The proof of this lemma can be found in the appendix.

LEMMA 2.6. *The sets $S^{(i)}_{(z_1, \ldots, z_n)}$ can be chosen in such a way that the fol-*

lowing holds:

(i) $S_i \cap S_j = \emptyset$ *if* $i \neq j$.

(ii) $S^{(i)}_{(z_1,\dots,z_n)} \cap \operatorname{supp}(\mu_j) = \emptyset$ *if* $z_j = s$ *and* $i \neq j$.

(iii) *Let* $i \neq j$ *and let* $(z_1,\dots,z_n) \in I_i$ *and* $(z_1',\dots,z_n') \in I_j$. *Suppose that* $z_i = o$, $z_j = a$ *and* $z_i' = a$, $z_j' = o$ *and* $z_l' = z_l$ *for* $l \neq i$ *and* $l \neq j$. *Then* $S^{(i)}_{(z_1,\dots,z_n)} = S^{(j)}_{(z_1',\dots,z_n')}$.

From now on, we assume that the sets $S^{(i)}_{(z_1,\dots,z_n)}$ are fixed and satisfy Lemma 2.6. Note that since it is assumed that $A_i \cap A_j = \emptyset$ if $i \neq j$, we have $A_i \subset S_i$. Note that the set S_i is such that $\mu_i(S_i) \geq 0$ and $\mu_j(S_i) = 0$ if $j \neq i$. We now state the main results of this paper.

THEOREM 2.7. *If* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$ *then*

$$\mathcal{MR}_k(\vec{\mu}) \text{ is convex}$$

$$\Leftrightarrow\ (k-1)\mu_i(\{x_m^{(i)}\}) \leq \mu_i(S_i \setminus A_i) + \sum_{j=m+1}^{\infty} \mu_i(\{x_j^{(i)}\}) \quad \forall i=1,\dots,n \ \forall m \geq 1$$

and if μ_i *is purely atomic then* A_i *is not a finite set.*

THEOREM 2.8. *If* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$ *then*

$$\mathcal{PR}(\vec{\mu}) \text{ is convex}$$

$$\Leftrightarrow\ \mu_i(\{x_m^{(i)}\}) \leq \mu_i(S_i \setminus A_i) + \sum_{j=m+1}^{\infty} \mu_i(\{x_j^{(i)}\}) \quad \forall i=1,\dots,n \ \forall m \geq 1$$

and if μ_i *is purely atomic then* A_i *is not a finite set.*

The proofs of Theorem 2.7 and Theorem 2.8 can be found in Section 4 and Section 5, respectively. From now on, let $(*)$ denote the right hand side of the equivalence arrow in Theorem 2.7 and let $(**)$ denote the right hand side of the equivalence arrow in Theorem 2.8.

Since $\mathcal{R}(\vec{\mu})$ is convex if and only if $\mathcal{MR}_2(\vec{\mu})$ is convex, Theorem 2.7 gives the following corollary.

COROLLARY 2.9. *If* $A_i \cap A_j = \emptyset$ *for all* $i \neq j$ *then*

$$\mathcal{R}(\vec{\mu}) \text{ is convex}$$

$$\Leftrightarrow\ \mu_i(\{x_m^{(i)}\}) \leq \mu_i(S_i \setminus A_i) + \sum_{j=m+1}^{\infty} \mu_i(\{x_j^{(i)}\}) \quad \forall i=1,\dots,n \ \forall m \geq 1$$

and if μ_i *is purely atomic then* A_i *is not a finite set.*

The assumption that the measures do not have any atoms in common is essential as can be seen in the next two examples concerning $\mathcal{R}(\vec{\mu})$.

EXAMPLE 2.10. Let $n = 2$ and $(\Omega, \mathcal{F}) = ([0,2], \textit{Borel sets})$. Let λ be the Lebesgue measure. Define two measures μ_1 and μ_2 as follows:

$$\mu_1(\{0\}) = \tfrac{1}{4} \qquad\qquad \mu_2(\{1\}) = \tfrac{1}{8}$$

$$\mu_1 = \lambda \qquad \text{on } [\tfrac{3}{8},\tfrac{9}{8}] \quad \mu_2 = \lambda \qquad \text{on } [0,\tfrac{7}{8}].$$

Then $A_1 = \{0\}$, $A_2 = \{1\}$. We can take $S_1 = \{0\} \cup [\frac{7}{8}, 1) \cup (1, \frac{9}{8}]$ and $S_2 = \{1\} \cup (0, \frac{3}{8}]$. An easy calculation shows that $\mu_1(\{0\}) \le \mu_1(S_1 \setminus A_1)$ and $\mu_2(\{1\}) \le \mu_2(S_2 \setminus A_2)$ so condition (**) of Corollary 2.9 is satisfied and thus by the same theorem $\mathscr{R}(\vec{\mu})$ is convex. A picture of $\mathscr{R}(\vec{\mu})$ can be found in Figure 1.

In the next example, the measures of Example 2.10 are changed just a little bit so that they have a common atom. It turns out that $\mathscr{R}(\vec{\mu})$ is not convex, even though the conditions of Corollary 2.9 are satisfied.

EXAMPLE 2.11. Let $n = 2$ and $(\Omega, \mathscr{F}) = ([0,2], \textit{Borel sets})$. Let λ be the Lebesgue measure. Define two measures μ_1 and μ_2 as follows:

$$\mu_1(\{2\}) = \tfrac{1}{4} \qquad\qquad \mu_2(\{2\}) = \tfrac{1}{8}$$

$$\mu_1 = \lambda \qquad \text{on } [\tfrac{3}{8},\tfrac{9}{8}] \quad \mu_2 = \lambda \qquad \text{on } [0,\tfrac{7}{8}].$$

Take $S_1 = (\frac{7}{8}, \frac{9}{8}]$ and $S_2 = [0, \frac{3}{8})$, $A_1 = A_2 = \{2\}$. So condition (**) of Corollary 2.9 is again satisfied. In this case, the range is obviously not convex (see Figure 2). So we can see that if $A_i \cap A_j \neq \emptyset$ for some $i \neq j$, then $\mathscr{R}(\vec{\mu})$ does not need to be convex, even if condition (**) is satisfied.

3. ENLARGING THE SPACE

In the proof of Theorem 2.7 we have to show that $\mathscr{MR}_k(\vec{\mu})$ is convex and in the proof of Theorem 2.8 we have to show that $\mathscr{PR}(\vec{\mu})$ is convex. In order to do this, $\mathscr{MR}_k(\vec{\mu})$ and $\mathscr{PR}(\vec{\mu})$ are compared with $\mathscr{MR}_k(\vec{\nu})$ and $\mathscr{PR}(\vec{\nu})$, respectively, where $\vec{\nu}$ is a nonatomic vector measure.

In this section, the measure $\vec{\nu}$ we want to use is defined. In order to define $\vec{\nu}$ the space $(\Omega, \mathscr{F})$ needs to be enlarged. This can be done as follows.

Suppose $(\Omega, \mathscr{F})$ and $(\Omega', \mathscr{F}')$ are two measurable spaces, where Ω and Ω' are disjoint sets. Then a new measurable space $(\Omega'', \mathscr{F}'')$ can be constructed, by let-

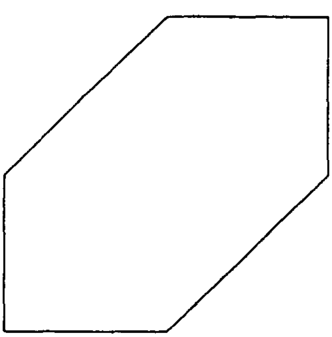

Figure 1

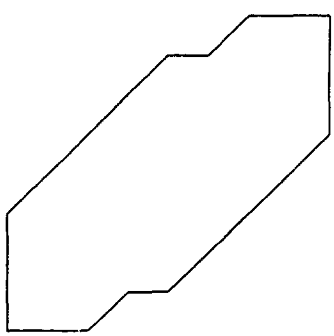

Figure 2

ting $\Omega'':=\Omega\cup\Omega'$ and $\mathscr{F}'':=\{F\cup F':F\in\mathscr{F},F'\in\mathscr{F}'\}$. It is easy to check that $\mathscr{F}''$ is a σ-algebra on Ω''. (Note that $(F\cup F')^c=(\Omega\setminus F)\cup(\Omega'\setminus F')$ if $F\in\mathscr{F}$ and $F'\in\mathscr{F}'$, since $\Omega\cap\Omega'=\emptyset$.)

Now suppose μ is a measure on $(\Omega,\mathscr{F})$. Then μ can be extended to a measure $\hat{\mu}$ on $(\Omega'',\mathscr{F}'')$ by letting $\hat{\mu}(F\cup F')=\mu(F)$ if $F\in\mathscr{F}$ and $F'\in\mathscr{F}'$.

This idea will be applied as follows. Suppose $\mu_1,\dots,\mu_n$ are measures on a measurable space $(\Omega,\mathscr{F})$. Recall that A_i is the set of atoms of μ_i and S_i is such that $\mu_i(S_i)\geq 0$ and $\mu_j(S_i)=0$ if $j\neq i$. Define $D_i=\Omega\setminus S_i$, and define new measures ϱ_i on $(\Omega,\mathscr{F})$, by letting $\varrho_i(F)=\mu_i(F\cap D_i)$, $i=1,\dots,n$. (So ϱ_i is the trace of μ_i in D_i.)

Now extend these measures as follows. Let $\Omega'=\mathbb{R}$ and let $\mathscr{F}'$ be the collection of Borel sets in $\mathbb{R}$. We consider Ω and Ω' to be disjoint. (Also if Ω itself is some subset of $\mathbb{R}$. This can always be achieved by labeling: replace Ω by $\{1\}\times\Omega$ and Ω' by $\{2\}\times\Omega'$. For notational convenience however, we just write Ω and $\Omega'(=\mathbb{R})$ and assume that the sets are disjoint.)

Then on the space $(\Omega'',\mathscr{F}'')$, define measures $\hat{\varrho}_i$ by $\hat{\varrho}_i(F\cup F'):=\varrho_i(F)$ for $F\in\mathscr{F}$, $F'\in\mathscr{F}'$, $i=1,\dots,n$.

Now we also follow a likewise procedure in the other direction. Let λ be the Lebesgue measure on the real line (on $(\Omega',\mathscr{F}')$). For $i=1,\dots,n$ define sets D_i' by $D_i':=[r_i,r_i+\mu_i(S_i)]$, where $r_1,\dots,r_n\in\mathbb{R}$ are chosen in such a way that $D_i'\cap D_j'=\emptyset$ if $i\neq j$ (this can be obtained by letting $r_1=0$, and $r_{i+1}=r_i+\mu_i(\Omega)+1$).

Define measures $\lambda_1,\dots,\lambda_n$ on $(\Omega',\mathscr{F}')$ by $\lambda_i(F'):=\lambda(F'\cap D_i')$, $F'\in\mathscr{F}'$ (so λ_i is the trace of λ in D_i' $i=1,\dots,n$). We can extend λ_i to a measure $\hat{\lambda}_i$ on $(\Omega'',\mathscr{F}'')$ by letting $\hat{\lambda}_i(F\cup F')=\lambda_i(F')$ for $F\in\mathscr{F}$, $F'\in\mathscr{F}'$, $i=1,\dots,n$.

Now finally define the measures $\nu_1,\dots,\nu_n$ on $(\Omega'',\mathscr{F}'')$ by

$$\nu_i=\hat{\varrho}_i+\hat{\lambda}_i$$

for $i=1,\dots,n$. Note that $\nu_i(\Omega'')=\mu_i(\Omega)$, $i=1,\dots,n$.

4. PROOF OF THEOREM 2.7

In this section, we prove Theorem 2.7. Before stating the proof, a few lemmas are needed.

LEMMA 4.1. *If $\mathscr{MR}_k(\vec{\mu})$ is convex, then $\mathscr{MR}_k(\mu_1),\dots,\mathscr{MR}_k(\mu_n)$ are convex.*

PROOF. Suppose $\mathscr{MR}_k(\vec{\mu})$ is convex. It will be shown that $\mathscr{MR}_k(\mu_1)$ is convex, $\mathscr{MR}_k(\mu_2),\dots,\mathscr{MR}_k(\mu_n)$ are convex by the same argument.

Let $(p_1,\dots,p_n)$ and $(q_1,\dots,q_n)\in\mathscr{MR}_k(\mu_1)$ and fix $0\leq\gamma\leq 1$. We have to show that $\gamma(p_1,\dots,p_n)+(1-\gamma)(q_1,\dots,q_n)\in\mathscr{MR}_k(\mu_1)$. Now there exist partitions $\{P_j\}_{j=1}^k$ and $\{Q_j\}_{j=1}^k$ such that $\mu_1(P_j)=p_j$ and $\mu_1(Q_j)=q_j$ for $j=1,\dots,k$. But then $M_1=(\mu_i(P_j))_{i=1}^{n}{}_{j=1}^{k}$ and $M_2=(\mu_i(Q_j))_{i=1}^{n}{}_{j=1}^{k}$ are in $\mathscr{MR}_k(\vec{\mu})$. Since $\mathscr{MR}_k(\vec{\mu})$ is convex, it follows that $\gamma M_1+(1-\gamma)M_2\in\mathscr{MR}_k(\vec{\mu})$. Hence there exists a partition $\{T_j\}_{j=1}^k$ such that $\mu_i(T_j)=\gamma\mu_i(P_j)+(1-\gamma)\mu_i(Q_j)$ for $i=1,\dots,n$, $j=1,\dots,k$. Since this holds in particular when $i=1$ it follows that $\mathscr{MR}_k(\mu_1)$ is convex. □

Now we state two more lemmas. They compare $\mathcal{MR}_k(\vec{\mu})$ with $\mathcal{MR}_k(\vec{\nu})$, where $\vec{\nu}$ is defined as in Section 3.

LEMMA 4.2. *Let $\mu_1, \ldots, \mu_n$ be measures satisfying $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ be defined as in Section 3. Then $\mathcal{MR}_k(\vec{\mu}) \subset \mathcal{MR}_k(\vec{\nu})$.*

PROOF. Let $(p_{ij})_{i=1}^{n}{}_{j=1}^{k} \in \mathcal{MR}(\vec{\mu})$. Then there exists a measurable partition $P_1, \ldots, P_k$ of Ω satisfying $\mu_i(P_j) = p_{ij}$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$. We have to find a partition $Q_1, \ldots, Q_k$ of Ω'' such that $\nu_i(Q_j) = p_{ij}$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$. Recall that $D_i = \Omega \setminus S_i$ and put $D = \bigcap_{i=1}^{n} D_i$. Write

$$P_j = (P_j \cap D) \cup \Big(\dot{\bigcup_{i=1}^{n}} (P_j \cap D_i^c)\Big) = (P_j \cap D) \cup \Big(\dot{\bigcup_{i=1}^{n}} (P_j \cap S_i)\Big).$$

It follows from the definition of ν_i that $\nu_i(P_j \cap D) = \mu_i(P_j \cap D)$. Note that $\mu_i(P_j \cap S_l) = 0$ if $l \neq i$ and $\mu_i(P_j \cap S_i) \geq 0$. For every i, $\{P_j \cap S_i\}_{j=1}^{k}$ is a measurable partition of S_i, with $\mu_i(\bigcup_{j=1}^{k} (P_j \cap S_i)) = \mu_i(S_i) = \nu_i(D_i')$. Since ν_i is non-atomic on D_i', there exists a partition $T_1^{(i)}, \ldots, T_k^{(i)}$ of D_i', satisfying $\nu_i(T_j^{(i)}) = \mu_i(P_j \cap S_i)$ (the existence of such a partition $T_1^{(i)}, \ldots, T_k^{(i)}$ can be obtained by repeatedly applying Lyapounov's convexity theorem [5]), and this can be done for every $i = 1, \ldots, n$. Now define (where $j = 1, \ldots, k-1$)

$$Q_j = (P_j \cap D) \cup \dot{\bigcup_{l=1}^{n}} T_j^{(l)}$$

$$Q_k = (P_k \cap D) \cup \dot{\bigcup_{l=1}^{n}} T_k^{(l)} \cup D^c \cup \Big(\Omega' \setminus \dot{\bigcup_{l=1}^{n}} D_l'\Big).$$

Then it is easy to check that $Q_1, \ldots, Q_k$ is a partition of Ω''. Also

$$\begin{aligned}
\nu_i(Q_j) &= \nu_i(P_j \cap D) + \sum_{l=1}^{n} \nu_i(T_j^{(l)}) \\
&= \mu_i(P_j \cap D) + \nu_i(T_j^{(i)}) \\
&= \mu_i(P_j \cap D) + \mu_i(P_j \cap D_i^c) \\
&= \mu_i(P_j \cap D) + \mu_i(P_j \cap D^c) \\
&= \mu_i(P_j) = p_{ij}
\end{aligned}$$

for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. So $(p_{ij})_{i=1}^{n}{}_{j=1}^{k} \in \mathcal{MR}_k(\vec{\nu})$ and we may conclude that $\mathcal{MR}_k(\vec{\mu}) \subset \mathcal{MR}_k(\vec{\nu})$. □

LEMMA 4.3. *Let $\mu_1, \ldots, \mu_n$ be measures satisfying $(*)$ and $A_i \cap A_j = \emptyset$ if $i \neq j$. Let $\vec{\nu} = (\nu_1, \ldots, \nu_n)$ be defined as in Section 3. Then $\mathcal{MR}_k(\vec{\mu}) \supset \mathcal{MR}_k(\vec{\nu})$.*

PROOF. Let $(p_{ij})_{i=1}^{n}{}_{j=1}^{k} \in \mathcal{MR}_k(\vec{\nu})$. Then there exists a partition $P_1'', \ldots, P_k''$ of Ω'', with $\nu_i(P_j'') = p_{ij}$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$. We have to find a partition $Q_1, \ldots, Q_k$ of Ω such that $\mu_i(Q_j) = p_{ij}$ for $i = 1, \ldots, n$, $j = 1, \ldots, k$.

Let $P_j = P_j'' \cap \Omega$ and $P_j' = P_j'' \cap \Omega'$. First notice that

$$v_i(P_j) = v_i(P_j \cap D) = \mu_i(P_j \cap D)$$

and

$$v_i(P_j') = v_i(P_j' \cap D_i').$$

Now $\{P_j' \cap D_i'\}_{j=1}^k$ is a partition of D_i' with $v_i(P_j' \cap D_i') = \beta_j^{(i)}$, where $\beta_j^{(i)} \geq 0$ and $\sum_{j=1}^k \beta_j^{(i)} = v_i(D_i') = \mu_i(S_i)$. Since $\mu_i|_{S_i}$ satisfies the condition of Theorem 2.3, this theorem gives that $\mathcal{MR}_k(\mu_i|_{S_i})$ is convex. Now $\mu_i(S_i)\vec{e}_j \in \mathcal{MR}_k(\mu_i|_{S_i})$, where $\vec{e}_1, \dots, \vec{e}_k$ are the unit vectors in $\mathbb{R}^k$. Write $\beta_j^{(i)} = \gamma_j^{(i)} \mu_i(S_i)$ for some $\gamma_j^{(i)} \geq 0$, and note that $\gamma_1^{(i)} + \cdots + \gamma_k^{(i)} = 1$. The convexity of $\mathcal{MR}_k(\mu_i|_{S_i})$ implies that $\sum_{j=1}^k \gamma_j^{(i)} \mu_i(S_i)\vec{e}_j = \sum_{j=1}^k \beta_j^{(i)} \vec{e}_j \in \mathcal{MR}_k(\mu_i|_{S_i})$ and thus there exists a partition $T_1^{(i)}, \dots, T_k^{(i)}$ of S_i satisfying $\mu_i(T_j^{(i)}) = \beta_j^{(i)} = v_i(P_j' \cap D_i')$. Define, for $j = 1, \dots, k$

$$Q_j = (P_j \cap D) \cup \bigcup_{l=1}^n T_j^{(l)}.$$

It is easy to see that $Q_1, \dots, Q_k$ is a partition of Ω and furthermore

$$\begin{aligned}
\mu_i(Q_j) &= \mu_i(P_j \cap D) + \sum_{l=1}^n \mu_i(T_j^{(l)}) \\
&= \mu_i(P_j \cap D) + \mu_i(T_j^{(i)}) \\
&= v_i(P_j \cap D) + v_i(P_j' \cap D_i') \\
&= v_i(P_j) + v_i(P_j') \\
&= v_i(P_j'') = p_{ij}
\end{aligned}$$

for $i = 1, \dots, n$ $j = 1, \dots, k$. So $(p_{ij})_{i=1}^n{}_{j=1}^k \in \mathcal{MR}_k(\vec{\mu})$ and thus we have proved that $\mathcal{MR}_k(\vec{v}) \subset \mathcal{MR}_k(\vec{\mu})$. $\square$

LEMMA 4.4. *Let $T \subset \operatorname{supp}(\mu_i)$, with $\mu_i(T) = 0$. Then $\mu_j(T) = 0$ for all $j \neq i$.*

PROOF. Without loss of generality, suppose that $i = 1$. Fix $j \neq 1$. Then

$$\mu_j(T) = \sum_{(z_1, \dots, z_n) \in I_j} \mu_j(T \cap S_{(z_1, \dots, z_n)}^{(j)}).$$

Of course, $\mu_1(T \cap S_{(z_1, \dots, z_n)}^{(j)}) = 0$ for all $(z_1, \dots, z_n) \in I_j$. We will prove that $\mu_j(T \cap S_{(z_1, \dots, z_n)}^{(j)}) = 0$ for all $(z_1, \dots, z_n) \in I_j$. Note that on the set $S_{(z_1, \dots, z_n)}^{(j)}$, μ_j equals $\sigma_{(z_1, \dots, z_n)}^{(j)}$. Now distinguish two cases: $z_1 = a$ and $z_1 = s$.

If $z_1 = a$, then $\sigma_{(z_1, \dots, z_n)}^{(j)} \ll \mu_1$ and thus $\sigma_{(z_1, \dots, z_n)}^{(j)}(T \cap S_{(z_1, \dots, z_n)}^{(j)}) = 0$. This implies $\mu_j(T \cap S_{(z_1, \dots, z_n)}^{(j)}) = 0$.

If $z_1 = s$, then $\sigma_{(z_1, \dots, z_n)}^{(j)} \perp \mu_1$. But in that case $S_{(z_1, \dots, z_n)}^{(j)} \cap \operatorname{supp}(\mu_1) = \emptyset$ by Lemma 2.6(ii). So $T \cap S_{(z_1, \dots, z_n)}^{(j)} = \emptyset$ and thus $\mu_j(T \cap S_{(z_1, \dots, z_n)}^{(j)}) = 0$. $\square$

Now we are ready to prove Theorem 2.7.

PROOF OF THEOREM 2.7. First we prove "$\Leftarrow$". Suppose that condition $(*)$ is satisfied. By Lemma 4.2 and Lemma 4.3, it follows that $\mathcal{MR}_k(\vec{\mu}) = \mathcal{MR}_k(\vec{\nu})$, with $\vec{\nu}$ as defined in Section 3. But $\vec{\nu}$ is nonatomic and hence by Theorem 1.5 $\mathcal{MR}_k(\vec{\nu})$ is convex. So $\mathcal{MR}_k(\vec{\mu})$ must also be convex. This completes the proof of "$\Leftarrow$".

Now we have to prove "$\Rightarrow$". Suppose condition ($*$) is not satisfied. We have to prove that $\mathcal{MR}_k(\vec{\mu})$ is not convex. There are two cases.

Case 1: there is a μ_i which is purely atomic and has a finite number of atoms.

Case 2: there is an $i \in \{1, \dots, n\}$ and an $m \in \mathbb{N}$ such that $(k-1)\mu_i(\{x_m^{(i)}\}) > \mu_i(S_i \setminus A_i) + \sum_{j=m+1}^{\infty} \mu_i(\{x_j^{(i)}\})$.

Case 1. Suppose without loss of generality that μ_1 is purely atomic with a finite number of atoms. Then $\mathcal{MR}_k(\mu_1)$ is not convex by Theorem 2.3 and thus by Lemma 4.1 $\mathcal{MR}_k(\vec{\mu})$ is not convex.

Case 2. Suppose without loss of generality that $i=1$. So there exists an $m \in \mathbb{N}$ satisfying

$$(k-1)\mu_1(\{x_m^{(1)}\}) > \mu_1(S_1 \setminus A_1) + \sum_{j=m+1}^{\infty} \mu_1(\{x_j^{(1)}\}).$$

Now $\mu_1|_{S_1}$ does not satisfy the conditions of Theorem 2.3, so $\mathcal{MR}_k(\mu_1|_{S_1})$ is not convex. Hence there exist $(a_1, \dots, a_k)$ and $(b_1, \dots, b_k) \in \mathcal{MR}_k(\mu_1|_{S_1})$ such that $\gamma(a_1, \dots, a_k) + (1-\gamma)(b_1, \dots, b_k) \notin \mathcal{MR}_k(\mu_1|_{S_1})$ for some $0 < \gamma < 1$. Fix that γ.

Now define two matrices $(p_{ij})_{i=1}^{n}{}_{j=1}^{k}$ and $(q_{ij})_{i=1}^{n}{}_{j=1}^{k}$ as follows. Let

$$\begin{aligned}
p_{1j} &= a_j && \text{for } j = 1, \dots, k-1 \\
p_{1k} &= a_k + \mu_1(\Omega \setminus S_1) && \\
p_{ij} &= 0 && \text{for } i = 2, \dots, n \text{ and } j = 1, \dots, k-1 \\
p_{ik} &= \mu_i(\Omega) && \text{for } i = 2, \dots, n
\end{aligned}$$

and

$$\begin{aligned}
q_{1j} &= b_j && \text{for } j = 1, \dots, k-1 \\
q_{1k} &= b_k + \mu_1(\Omega \setminus S_1) && \\
q_{ij} &= 0 && \text{for } i = 2, \dots, n \text{ and } j = 1, \dots, k-1 \\
q_{ik} &= \mu_i(\Omega) && \text{for } i = 2, \dots, n.
\end{aligned}$$

Let $P_1, \dots, P_k$ be a partition as follows. Let $P_1', \dots, P_k'$ be a partition of S_1 satisfying $\mu_1(P_j') = a_j$, $j = 1, \dots, k$. Then let

$$\begin{aligned}
P_j &= P_j' && \text{for } j = 1, \dots, k-1 \\
P_k &= P_k' \cup (\Omega \setminus S_1). &&
\end{aligned}$$

Then it is easy to see that $\mu_i(P_j) = p_{ij}$, $i = 1, \dots, n$, $j = 1, \dots, k$. So $(p_{ij})_{i=1}^{n}{}_{j=1}^{k} \in \mathcal{MR}_k(\vec{\mu})$.

Similarly, let $Q_1, \dots, Q_k$ be a partition as follows. Let $Q_1', \dots, Q_k'$ be a partition of S_1 satisfying $\mu_1(Q_j') = b_j$, $j = 1, \dots, k$. Then let

$$Q_j = Q_j' \qquad \text{for } j=1,\dots,k-1$$

$$Q_k = Q_k' \cup (\Omega \setminus S_1).$$

Then it is easy to see that $\mu_i(Q_j) = q_{ij}$, $i=1,\dots,n$, $j=1,\dots,k$. So $(q_{ij})_{i=1}^{n}\,{}_{j=1}^{k} \in \mathcal{MR}_k(\vec{\mu})$.

Define $(t_{ij})_{i=1}^{n}\,{}_{j=1}^{k}$ by $t_{ij} = \gamma p_{ij} + (1-\gamma)q_{ij}$ $i=1,\dots,n$ $j=1,\dots,k$. We will show that $(t_{ij})_{i=1}^{n}\,{}_{j=1}^{k} \notin \mathcal{MR}_k(\vec{\mu})$ and hence $\mathcal{MR}_k(\vec{\mu})$ is not convex. Note that

$$t_{1j} = \gamma a_j + (1-\gamma) b_j \qquad \text{for } j=1,\dots,k-1$$

$$t_{1k} = \gamma a_k + (1-\gamma) b_k + \mu_1(\Omega \setminus S_1)$$

$$t_{ij} = 0 \qquad \text{for } i=2,\dots,n \text{ and } j=1,\dots,k-1$$

$$t_{ik} = \mu_i(\Omega) \qquad \text{for } i=2,\dots,n.$$

It will be shown that there does not exist a partition $T_1,\dots,T_k$ of Ω such that $\mu_i(T_j) = t_{ij}$, $i=1,\dots,n$, $j=1,\dots,k$ and hence $(t_{ij})_{i=1}^{n}\,{}_{j=1}^{k} \notin \mathcal{MR}_k(\vec{\mu})$. Arguing by contradiction, suppose such a partition exists. Since $\mu_i(T_k) = \mu_i(\Omega)$ $(i=2,\dots,k)$, we know that $T_k \supset (\Omega \setminus S_1)$. For

$$T_k \supset \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\} \quad \text{for } i=2,\dots,n$$

(modulo a set of μ_i-measure 0, but a subset of $\bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\} = \operatorname{supp}(\mu_i)$, which has μ_i-measure 0 also has μ_j-measure 0 for $j \neq i$ by Lemma 4.4). Thus

$$T_k \supset \bigcup_{i=2}^{n} \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\}.$$

Lemma 2.6(iii), gives that for $i \geq 2$:

$$S^{(i)}_{(a,z_2,\dots,z_{i-1},o,z_{i+1},\dots,z_n)} = S^{(1)}_{(o,z_2,\dots,z_{i-1},a,z_{i+1},\dots,z_n)},$$

and thus it follows that

$$\bigcup_{i=2}^{n} \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\} = \Big(\bigcup_{i=1}^{n} \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\}\Big) \setminus S_1.$$

Since $\mu_i(\Omega \setminus (\bigcup_{i=1}^{n} \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\})) = 0$ for all $i=1,\dots,n$, it follows that without loss of generality $T_k \supset \Omega \setminus S_1$.

So we have to find a partition $T_1',\dots,T_k'$ of S_1 such that

$$\mu_1(T_j') = \gamma a_j + (1-\gamma) b_j, \quad j=1,\dots,k.$$

Then $T_j = T_j'$, $j=1,\dots,k-1$ and $T_k = T_k' \cup (\Omega \setminus S_1)$ would give us the desired partition.

But it is impossible to find such a partition $T_1',\dots,T_k'$ since $\gamma(a_1,\dots,a_k) + (1-\gamma)(b_1,\dots,b_k) \notin \mathcal{MR}_k(\mu_1|_{S_1})$. So there does not exist a partition $T_1,\dots,T_k$ of Ω such that $\mu_i(T_j) = t_{ij}$ $i=1,\dots,n$ $j=1,\dots,k$ and hence $\mathcal{MR}_k(\vec{\mu})$ is not convex. This completes the proof of case 2 and thus "$\Rightarrow$" is proved. □

REMARK. Note that if the measures are nonatomic, then condition $(*)$ is trivially satisfied and thus $\mathcal{MR}_k(\vec{\mu})$ is convex. To Theorem 2.7 is indeed a generalization of Theorem 1.5.

5. THE PARTITION-RANGE

In this section, we take a closer look at the partition-range. Note that convexity of $\mathcal{MR}_n(\vec{\mu})$ implies convexity of $\mathcal{PR}(\vec{\mu})$. But condition (**) is a weaker condition than (*) with $k=n$. In this section, Theorem 2.8 will be proved. First some tools are developed. Note that Theorem 1.5 implies the following theorem.

THEOREM 5.1. *If $\vec{\mu}$ is nonatomic, then $\mathcal{PR}(\vec{\mu})$ is convex.*

We also need the following:

LEMMA 5.2. *If $\mathcal{PR}(\vec{\mu})$ is convex, then $\mathcal{R}(\mu_1),\dots,\mathcal{R}(\mu_n)$ are convex.*

PROOF. Suppose that $\mathcal{PR}(\vec{\mu})$ is convex. We will prove that $\mathcal{R}(\mu_1)$ is convex, $\mathcal{R}(\mu_2),\dots,\mathcal{R}(\mu_n)$ are convex by the same argument.

We have to show that $\mathcal{R}(\mu_1)=[0,\mu_1(\Omega)]$. Clearly $\mathcal{R}(\mu_1)\subset[0,\mu_1(\Omega)]$ always holds, so it remains to show that $\mathcal{R}(\mu_1)\supset[0,\mu_1(\Omega)]$.

Let $\alpha\in[0,\mu_1(\Omega)]$ be arbitrary. It must be shown that $\alpha\in\mathcal{R}(\mu_1)$. Write $\alpha=\gamma\mu_1(\Omega)$ for some $0\le\gamma\le 1$. Let $\vec{e}_1,\dots,\vec{e}_n$ denote the unit vectors in $\mathbb{R}^n$. Then, by taking the partition $P_i=\Omega$ and $P_j=\emptyset$ if $j\neq i$, one can easily see that $\mu_i(\Omega)\vec{e}_i\in\mathcal{PR}(\vec{\mu})$ $i=1,\dots,n$.

Now $\mathcal{PR}(\vec{\mu})$ is convex, so $\gamma\mu_1(\Omega)\vec{e}_1+(1-\gamma)\mu_2(\Omega)\vec{e}_2\in\mathcal{PR}(\vec{\mu})$. Thus, there exists a partition $Q_1,\dots,Q_n$ of Ω such that $\mu_1(Q_1)=\gamma\mu_1(\Omega)$, $\mu_2(Q_2)=(1-\gamma)\mu_2(\Omega)$ and $\mu_i(Q_i)=0$ if $i=3,\dots,n$. But then there exists a set Q_1 such that $\mu_1(Q_1)=\gamma\mu_1(\Omega)=\alpha$, so $\alpha\in\mathcal{R}(\mu_1)$. So $\mathcal{R}(\mu_1)=[0,\mu_1(\Omega)]$ and thus $\mathcal{R}(\mu_1)$ is convex. □

Next we will compare $\mathcal{PR}(\vec{\mu})$ with $\mathcal{PR}(\vec{\nu})$, where $\vec{\nu}$ is defined as in Section 3.

LEMMA 5.3. *Let $\mu_1,\dots,\mu_n$ be measures satisfying $A_i\cap A_j=\emptyset$ if $i\neq j$, and let $\vec{\nu}=(\nu_1,\dots,\nu_n)$ be defined as in Section 3. Then $\mathcal{PR}(\vec{\mu})\subset\mathcal{PR}(\vec{\nu})$.*

PROOF. Let $\vec{p}=(p_1,\dots,p_n)\in\mathcal{PR}(\vec{\mu})$. Then there exists a partition $P_1,\dots,P_n$ of Ω satisfying $\mu_i(P_i)=p_i$ for $i=1,\dots,n$. We have to construct a partition $Q_1,\dots,Q_n$ of Ω'' such that $\nu_i(Q_i)=p_i$ for $i=1,\dots,n$.

Recall from Section 3 that $D_i=\Omega\setminus S_i$. Let

$$D=\bigcap_{i=1}^{n}D_i=\bigcap_{i=1}^{n}(\Omega\setminus S_i)=\Omega\setminus(\bigcup_{i=1}^{n}S_i).$$

We can write $P_i=(P_i\cap D)\cup(P_i\cap D^c)$ for $i=1,\dots,n$. Now $P_i\cap D\subset P_i\cap D_i$ and by definition $\nu_i\,|_{D_i}=\mu_i\,|_{D_i}$, so $\nu_i(P_i\cap D)=\mu_i(P_i\cap D)$ (for $i=1,\dots,n$).

Furthermore

$$P_i\cap D^c=P_i\cap(\bigcap_{j=1}^{n}D_j)^c=\bigcup_{j=1}^{n}(P_i\cap D_j^c)=\bigcup_{j=1}^{n}(P_i\cap S_j).$$

Since $\mu_i(P_i \cap S_j) \le \mu_i(S_j) = 0$ if $j \ne i$, it follows that $0 \le \mu_i(P_i \cap D^c) = \mu_i(P_i \cap S_i) \le \mu_i(S_i)$.

Now consider ν_i. As $\nu_i|_{D_i'}$ is nonatomic, Lyapounov's Theorem gives that $\mathscr{R}(\nu_i|_{D_i'})$ is convex. Hence $\mathscr{R}(\nu_i|_{D_i'}) = [0, \nu_i(D_i')] = [0, \mu_i(S_i)]$. But then there exists a set $T_i \subset D_i'$ satisfying $\nu_i(T_i) = \mu_i(P_i \cap D^c)$. Since $D_i' \cap D_j' = \emptyset$ if $i \ne j$, it follows that $\nu_j(T_i) = 0$ if $j \ne i$.

Now define the following sets (where $i = 1, \dots, n-1$):

$$Q_i = (P_i \cap D) \cup T_i \cup S_{i+1} \cup (D_{i+1}' \setminus T_{i+1})$$

$$Q_n = (P_n \cap D) \cup T_n \cup S_1 \cup (D_1' \setminus T_1') \cup (\Omega' \setminus \bigcup_{j=1}^{n} D_j').$$

Then is is easy to check that $Q_1, \dots, Q_n$ is a measurable partition of Ω'' and

$$\begin{aligned}\nu_i(Q_i) &= \nu_i(P_i \cap D) + \nu_i(T_i)\\ &= \mu_i(P_i \cap D) + \mu_i(P_i \cap D^c)\\ &= \mu_i(P_i) = p_i.\end{aligned}$$

So $\vec{p} = (p_1, \dots, p_n) \in \mathscr{PR}(\vec{\nu})$ and thus $\mathscr{PR}(\vec{\mu}) \subset \mathscr{PR}(\vec{\nu})$. □

LEMMA 5.4. *Let $\mu_1, \dots, \mu_n$ be measures satisfying condition $(**)$ and $A_i \cap A_j = \emptyset$ if $i \ne j$, and let $\vec{\nu} = (\nu_1, \dots, \nu_n)$ be defined as in Section 3. Then $\mathscr{PR}(\vec{\mu}) \supset \mathscr{PR}(\vec{\nu})$.*

PROOF. Let $\vec{p} = (p_1, \dots, p_n) \in \mathscr{PR}(\vec{\nu})$. Then there exists a measurable partition $P_1'', \dots, P_n''$ of Ω'' such that $\nu_i(P_i'') = p_i$, $i = 1, \dots, n$. We have to construct a partition $Q_1, \dots, Q_n$ of Ω such that $\mu_i(Q_i) = p_i$, $i = 1, \dots, n$. Write $P_i'' = P_i \cup P_i'$, where $P_i = P_i'' \cap \Omega$ and $P_i' = P_i'' \cap \Omega'$. Notice that $\nu_i(P_i) = \nu_i(P_i \cap D_i)$. Also $P_i \cap D_i = (P_i \cap D) \cup \bigcup_{j \ne i} (P_i \cap D_i \cap D_j^c)$. Using the fact that $\mu_i(D_j^c) = \mu_i(S_j) = 0$ if $i \ne j$, it follows that

$$\begin{aligned}\nu_i(P_i) &= \nu_i(P_i \cap D_i) = \mu_i(P_i \cap D_i)\\ &= \mu_i(P_i \cap D) + \sum_{j \ne i} \mu_i(P_i \cap D_i \cap D_j^c) = \mu_i(P_i \cap D),\end{aligned}$$

for $i = 1, \dots, n$.

Next we know that $\nu_i(P_i') = \nu_i(P_i' \cap D_i') \in [0, \nu_i(D_i')] = [0, \mu_i(S_i)]$. But $\mu_i|_{S_i}$ satisfies the condition of Corollary 2.9 (with $n = 1$), so $\mathscr{R}(\mu_i|_{S_i})$ is convex and hence $\mathscr{R}(\mu_i|_{S_i}) = [0, \mu_i(S_i)]$, $i = 1, \dots, n$. This implies the existence of a set $T_i \subset S_i$ $(= D_i^c)$ such that $\mu_i(T_i) = \nu_i(P_i' \cap D_i')$, $i = 1, \dots, n$. Now construct $Q_1, \dots, Q_n$ as follows (where $i = 1, \dots, n-1$):

$$Q_i = (P_i \cap D) \cup T_i \cup (S_{i+1} \setminus T_{i+1})$$

$$Q_n = (P_n \cap D) \cup T_n \cup (S_1 \setminus T_1).$$

Then it is easy to check that $Q_1, \dots, Q_n$ is indeed a partition of Ω and

$$\begin{aligned}\mu_i(Q_i) &= \mu_i(P_i \cap D) + \mu_i(T_i)\\ &= \nu_i(P_i) + \nu_i(P_i' \cap D_i')\end{aligned}$$

$$= \nu_i(P_i) + \nu_i(P_i')$$
$$= \nu_i(P_i'') = p_i$$

for $i=1,\dots,n$. So $\vec{p}=(p_1,\dots,p_n)\in\mathscr{PR}(\vec{\mu})$ and thus $\mathscr{PR}(\vec{\nu})\subset\mathscr{PR}(\vec{\mu})$. $\square$

Now we are ready to prove Theorem 2.8.

PROOF OF THEOREM 2.8. First we prove that "$\Rightarrow$" holds. Suppose that $(**)$ is not satisfied. We have to prove that $\mathscr{PR}(\vec{\mu})$ is not convex. Consider two cases.

Case 1: one of the measures is purely atomic with a finite number of atoms.

Case 2: there exists an $i\in\{1,\dots,n\}$ and there exists an $m\in\mathbb{N}$ such that $\mu_i(\{x_m^{(i)}\})>\mu_i(S_i\setminus A_i)+\sum_{j=m+1}^{\infty}\mu_i(\{x_j^{(i)}\})$.

Case 1. Without loss of generality assume that μ_1 is purely atomic with a finite number of atoms. Then $\mathscr{R}(\mu_1)$ is not convex by Corollary 2.9 and hence by Lemma 5.2, $\mathscr{PR}(\vec{\mu})$ is not convex.

Case 2. Without loss of generality suppose $i=1$. So there exists an $m\in\mathbb{N}$ with $\mu_1(\{x_m^{(1)}\})>\mu_1(S_1\setminus A_1)+\sum_{j=m+1}^{\infty}\mu_1(\{x_j^{(1)}\})$. Construct two partitions as follows.

$$P_1 = \{x_m^{(1)}\}$$
$$P_2 = (S_1\setminus\{x_m^{(1)}\})\cup\left(\bigcup\{S^{(2)}_{(z_1,\dots,z_n)}\mid(z_1,\dots,z_n)\in I_2\}\right)$$
$$\cup\left(\Omega\setminus\left(\bigcup_{i=1}^{n}\bigcup\{S^{(i)}_{(z_1,\dots,z_n)}\mid(z_1,\dots,z_n)\in I_i\}\right)\right)$$
$$P_k = \bigcup\{S^{(k)}_{(z_1,\dots,z_n)}\mid(z_1,\dots,z_n)\in I_k;\ z_2=\cdots=z_{k-1}=s\}\quad\text{for } k=3,\dots,n.$$

$$Q_1 = S_1\setminus\{x_1^{(1)},\dots,x_m^{(1)}\}$$
$$Q_2 = \{x_1^{(1)},\dots,x_m^{(1)}\}\cup\left(\bigcup\{S^{(2)}_{(z_1,\dots,z_n)}\mid(z_1,\dots,z_n)\in I_2\}\right)$$
$$\cup\left(\Omega\setminus\left(\bigcup_{i=1}^{n}\bigcup\{S^{(i)}_{(z_1,\dots,z_n)}\mid(z_1,\dots,z_n)\in I_i\}\right)\right)$$
$$Q_k = P_k\quad\text{for } k=3,\dots,n.$$

After some calculations using Lemma 2.6, it follows that $P_1,\dots,P_n$ and $Q_1,\dots,Q_n$ are indeed partitions, so $(\mu_1(P_1),\dots,\mu_n(P_n))$ and $(\mu_1(Q_1),\dots,\mu_n(Q_n))$ are in $\mathscr{PR}(\vec{\mu})$. Note that $\mu_2(P_2)=\mu_2(Q_2)=\mu_2(\Omega)$. We will prove that $\mathscr{PR}(\vec{\mu})$ is not convex by showing that $\frac{1}{2}(\mu_1(P_1),\dots,\mu_n(P_n))+\frac{1}{2}(\mu_1(Q_1),\dots,\mu_n(Q_n))\notin\mathscr{PR}(\vec{\mu})$.

Suppose there exists a partition $B_1,\dots,B_n$ satisfying

$$\mu_i(B_i)=\tfrac{1}{2}\mu_i(P_i)+\tfrac{1}{2}\mu_i(Q_i),\quad i=1,\dots,n.$$

In other words:

$$\mu_1(B_1)=\tfrac{1}{2}\mu_1(P_1)+\tfrac{1}{2}\mu_1(Q_1)$$
$$\mu_i(B_i)=\mu_i(P_i)=\mu_i(Q_i)\quad\text{for } i=2,\dots,n.$$

Note that

$$B_2 \supset \bigcup \{S^{(2)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_2\},$$

otherwise $\mu_2(B_2) < \mu_2(\Omega) = \mu_2(P_2)$. (A priori this only holds modulo a set of μ_2-measure 0, but we can say this because of Lemma 4.4. Note that

$$\bigcup \{S^{(2)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_2\} = \mathrm{supp}(\mu_2)$$

and a set $T \subset \mathrm{supp}(\mu_2)$ with μ_2-measure 0, has also μ_j-measure 0 for $j \neq 2$). Given the fact that

$$B_2 \supset \bigcup \{S^{(2)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_2\},$$

we must have

$$B_3 \supset \bigcup \{S^{(3)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_3;\ z_2 = s\}.$$

(Again modulo a set of μ_3-measure 0, but again by Lemma 4.4, this is of no importance). This argument can be continued. Given the fact that

$$B_{j-1} \supset \bigcup \{S^{(j-1)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_{j-1};\ z_2 = \cdots = z_{j-2} = s\},$$

we must have

$$B_j \supset \bigcup \{S^{(j)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_j;\ z_2 = \cdots = z_{j-1} = s\},$$

for $j = 3, \dots, n$.

So $(B_2 \cup \cdots \cup B_n) \supset \bigcup_{i=2}^n \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\}$. This implies

$$B_1 \subset \Omega \backslash (\bigcup_{i=2}^n \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\}).$$

Using Lemma 2.6(iii), we know that for $i \geq 2$, we have

$$S^{(i)}_{(a,z_2,\dots,z_{i-1},0,z_{i+1},\dots,z_n)} = S^{(1)}_{(0,z_2,\dots,z_{i-1},a,z_{i+1},\dots,z_n)}.$$

Therefore

$$\begin{aligned}&\Omega \backslash (\bigcup_{i=2}^n \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\})\\ &\quad \subset S_1 \cup \bigcup (\Omega \backslash (\bigcup_{i=1}^n \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\})).\end{aligned}$$

So $B_1 \subset S_1 \cup (\Omega \backslash (\bigcup_{i=1}^n \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\}))$. Since

$$\mu_1(\Omega \backslash (\bigcup_{i=1}^n \bigcup \{S^{(i)}_{(z_1,\dots,z_n)} \mid (z_1,\dots,z_n) \in I_i\})) = 0,$$

we can assume that $B_1 \subset S_1$.

First we claim that $\{x_1^{(1)}, \dots, x_m^{(1)}\} \cap B_1 = \emptyset$. For suppose not, then

$$\mu_1(B_1) \geq \mu_1(\{x_m^{(1)}\}) = \mu_1(P_1) > \tfrac{1}{2}\mu_1(P_1) + \tfrac{1}{2}\mu_1(Q_1) = \mu_1(B_1),$$

which gives a contradiction. So $B_1 \subset S_1 \backslash \{x_1^{(1)}, \dots, x_m^{(1)}\}$. But in that case

$$
\begin{aligned}
\mu_1(B_1) &\leq \mu_1(S_1 \setminus A_1) + \sum_{j=m+1}^{\infty} \mu_1(\{x_j^{(1)}\}) \\
&= \mu_1(Q_1) \\
&< \tfrac{1}{2}\mu_1(P_1) + \tfrac{1}{2}\mu_1(Q_1) \\
&= \mu_1(B_1)
\end{aligned}
$$

which gives a contradiction. But then it is impossible to construct such a partition $B_1, \ldots, B_n$ and hence $\mathscr{PR}(\vec{\mu})$ is not convex. This completes the proof of case 2 and thus we have proved "$\Rightarrow$".

It remains to show "$\Leftarrow$". Let $\nu_1, \ldots, \nu_n$ be defined as in Section 3. Then by Lemma 5.3 and Lemma 5.4, $\mathscr{PR}(\vec{\mu}) = \mathscr{PR}(\vec{\nu})$. But $\vec{\nu}$ is nonatomic, so $\mathscr{PR}(\vec{\nu})$ is convex by Theorem 5.1, and thus $\mathscr{PR}(\vec{\mu})$ is convex. This completes the proof of Theorem 2.8. $\square$

6. APPLICATIONS

The results of Section 4 and Section 5 can be applied to partitioning problems. In partitioning problems, we are looking for a measurable partition $P_1, \ldots, P_n$ of Ω such that $\mu_i(P_i)$ is bigger than some number (e.g. $1/n$ for all i). Many of these problems use the convexity of $\mathscr{MR}_k(\vec{\mu})$ or $\mathscr{PR}(\vec{\mu})$ in their proof. For example in Hill [4], the proof of the main theorem depends on the convexity of $\mathscr{MR}_k(\vec{\mu})$. The result is very general, but since it involves much notation, we will not state it here. Instead we will give a few other examples, which are presented here as corollaries from Theorem 2.7 and 2.8.

We start with applications of the convexity of $\mathscr{MR}_k(\vec{\mu})$. First a generalization of a result by Neyman [6] is stated.

COROLLARY 6.1. *If $\mu_1, \ldots, \mu_n$ are probability measures satisfying* (*) *and $A_i \cap A_j = \emptyset$ for $i \neq j$ then there exists a measurable k-partition $\{P_j\}_{j=1}^k$ of Ω satisfying $\mu_i(P_j) = 1/k$, $i = 1, \ldots, n$, $j = 1, \ldots, k$.*

PROOF. Define matrices $M_1, \ldots, M_k$ as follows: $M_l = (\mu_i(P_j^{(l)}))_{i=1}^{n}{}_{j=1}^{k}$, where $\{P_j^{(l)}\}_{j=1}^k$ is the partition of Ω with $P_l^{(l)} = \Omega$ and $P_j^{(l)} = \emptyset$ if $j \neq l$. Then M_l is the matrix of which the l-th column contains only 1's and all the other entries are 0. It is obvious that $M_1, \ldots, M_k \in \mathscr{MR}_k(\vec{\mu})$. Since $\mathscr{MR}_k(\vec{\mu})$ is convex, it follows that $(1/k)M_1 + \cdots + (1/k)M_k \in \mathscr{MR}_k(\vec{\mu})$. $\square$

The next result is proved by Dubins and Spanier [1] for nonatomic measures. The proof is almost the same as the proof of Corollary 6.1 and is therefore omitted.

COROLLARY 6.2. *If $\mu_1, \ldots, \mu_n$ are probability measures satisfying* (*) *and $A_i \cap A_j = \emptyset$ for $i \neq j$, and $\alpha_1, \ldots, \alpha_k \geq 0$ with $\sum_{j=1}^k \alpha_j = 1$, then there exists a measurable k-partition $\{P_j\}_{j=1}^k$ of Ω satisfying $\mu_i(P_j) = \alpha_j$ for $i = 1, \ldots, n$ and $j = 1, \ldots, k$.*

Now we give some applications of the convexity of $\mathscr{PR}(\vec{\mu})$.

COROLLARY 6.3. *Let $\mu_1,\dots,\mu_n$ be probability measures on $(\Omega,\mathscr{F})$ satisfying $(**)$ and $A_i\cap A_j=\emptyset$ if $i\neq j$. Let $\alpha_1,\dots,\alpha_n\in\mathbb{R}$ with $\alpha_i\geq 0$, $i=1,\dots,n$ and $\alpha_1+\cdots+\alpha_n=1$. Then there exists a measurable partition $P_1,\dots,P_n$ of Ω such that $\mu_i(P_i)=\alpha_i$.*

PROOF. We know $\vec{e}_i\in\mathscr{PR}(\vec{\mu})$, $i=1,\dots,n$ (where $\vec{e}_1,\dots,\vec{e}_n$ denote the unit vectors in $\mathbb{R}^n$). Since $(**)$ satisfied, $\mathscr{PR}(\vec{\mu})$ is convex by Theorem 2.8. Now $\alpha_i\geq 0$ for $i=1,\dots,n$ and $\alpha_1+\cdots+\alpha_n=1$, so by the convexity of $\mathscr{PR}(\vec{\mu})$ we have $\alpha_1\vec{e}_1+\cdots+\alpha_n\vec{e}_n\in\mathscr{PR}(\vec{\mu})$. Thus there exists a measurable partition $P_1,\dots,P_n$ of Ω such that $\mu_i(P_i)=\alpha_i$. $\square$

The next application is a generalization of a result by Dubins and Spanier [1]. The proof is almost literally the same as in [1]. Dubins and Spanier use convexity of the matrix-n-range in their proof, but they are only interested in the diagonal, which is exactly the partition-range. For the rest the proof can be copied and is therefore omitted here.

COROLLARY 6.4. *Let $\mu_1,\dots,\mu_n$ be probability measures on $(\Omega,\mathscr{F})$, satisfying $(**)$ and $A_i\cap A_j=\emptyset$ if $i\neq j$. Suppose $\mu_j\neq\mu_k$ for some $j\neq k$. Let $\alpha_i>0$, $i=1,\dots,n$ and $\alpha_1+\cdots+\alpha_n=1$. Then there exists a partition $P_1,\dots,P_n$ of Ω such that $\mu_i(P_i)>\alpha_i$.*

A. APPENDIX

In this appendix Theorem 2.5 and Lemma 2.6 will be proved.

THEOREM 2.5. *Let $\mu_1,\dots,\mu_n$ be measures. Let I_i be the following index set:*

$$I_i:=\{(z_1,\dots,z_n)\mid z_j\in\{a,s\}\text{ if } j\neq i \text{ and } z_i=o\}.$$

Then μ_i can be decomposed as follows:

$$\mu_i=\sum_{(z_1,\dots,z_n)\in I_i}\sigma^{(i)}_{(z_1,\dots,z_n)}$$

where for $j\neq i$

$$\sigma^{(i)}_{(z_1,\dots,z_n)}\ll\mu_j \quad \text{if } z_j=a$$

$$\sigma^{(i)}_{(z_1,\dots,z_n)}\perp\mu_j \quad \text{if } z_j=s.$$

A decomposition of μ_i with these properties is unique. Moreover, if $(z_1,\dots,z_n)$ and $(z'_1,\dots,z'_n)\in I_i$, and $(z_1,\dots,z_n)\neq(z'_1,\dots,z'_n)$, then $\sigma^{(i)}_{(z_1,\dots,z_n)}\perp\sigma^{(i)}_{(z'_1,\dots,z'_n)}$.

PROOF. We will show that such a decomposition exists for μ_1, for all the other measures we can use the same argument. For notational convenience, the upper index (1) will not be carried explicitly. The existence of such a decomposition follows, using Theorem 2.4.

We know that $\mu_1 = \sigma_{oa} + \sigma_{os}$, where $\sigma_{oa} \ll \mu_2$ and $\sigma_{os} \perp \mu_2$. Now σ_{oa} and σ_{os} are again measures, so there exists a decomposition of σ_{oa} and σ_{os} with respect to μ_3:

$$\sigma_{oa} = \sigma_{oaa} + \sigma_{oas}, \quad \text{where } \sigma_{oaa} \ll \mu_3 \text{ and } \sigma_{oas} \perp \mu_3$$

and

$$\sigma_{os} = \sigma_{osa} + \sigma_{oss}, \quad \text{where } \sigma_{osa} \ll \mu_3 \text{ and } \sigma_{oss} \perp \mu_3.$$

So $\mu_1 = \sigma_{oaa} + \sigma_{oas} + \sigma_{osa} + \sigma_{oss}$. On these four measures we can apply the same argument and make a decomposition with respect to μ_4. Using an iteration argument, we obtain the desired decomposition.

Now we have to show the uniqueness of the decomposition. Suppose

$$\mu_1 = \sum_{(z_1,\dots,z_n)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_n)\in I_1} \sigma'_{(z_1,\dots,z_n)}$$

are both decompositions which satisfy the conditions of the theorem. We will show that in that case we must have $\sigma_{(z_1,\dots,z_n)} = \sigma'_{(z_1,\dots,z_n)}$ for all $(z_1,\dots,z_n)\in I_1$.

Note that

$$\mu_1 = \sum_{(z_1,\dots,z_n)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} + \sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma_{(z_1,\dots,z_n)}$$

with $\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} \ll \mu_n$ and $\sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma_{(z_1,\dots,z_n)} \perp \mu_n$. Also

$$\mu_1 = \sum_{(z_1,\dots,z_n)\in I_1} \sigma'_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma'_{(z_1,\dots,z_n)} + \sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma'_{(z_1,\dots,z_n)}$$

with $\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma'_{(z_1,\dots,z_n)} \ll \mu_n$ and $\sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma'_{(z_1,\dots,z_n)} \perp \mu_n$.

Now apply the uniqueness part of Theorem 2.4. This gives us that

$$\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma'_{(z_1,\dots,z_n)}$$

$$\sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma'_{(z_1,\dots,z_n)}.$$

We can apply the same argument to $\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)}$ (and of course an analogous argument to $\sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma_{(z_1,\dots,z_n)}$):

$$\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma_{(z_1,\dots,z_n)} + \sum_{(z_1,\dots,z_{n-2},a,s)\in I_1} \sigma_{(z_1,\dots,z_n)}$$

with $\sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma_{(z_1,\dots,z_n)} \ll \mu_{n-1}$ and $\sum_{(z_1,\dots,z_{n-2},a,s)\in I_1} \sigma_{(z_1,\dots,z_n)} \perp \mu_{n-1}$.

Now since

$$\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma'_{(z_1,\dots,z_n)},$$

we also have

$$\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma'_{(z_1,\dots,z_n)} + \sum_{(z_1,\dots,z_{n-2},a,s)\in I_1} \sigma'_{(z_1,\dots,z_n)}$$

with $\sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma'_{(z_1,\dots,z_n)} \ll \mu_{n-1}$ and $\sum_{(z_1,\dots,z_{n-2},a,s)\in I_1} \sigma'_{(z_1,\dots,z_n)} \perp \mu_{n-1}$.

The uniqueness part of Theorem 2.4 gives us that

$$\sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma'_{(z_1,\dots,z_n)}$$

$$\sum_{(z_1,\dots,z_{n-2},a,s)\in I_1} \sigma_{(z_1,\dots,z_n)} = \sum_{(z_1,\dots,z_{n-2},a,s)\in I_1} \sigma'_{(z_1,\dots,z_n)}.$$

By repeating this argument over and over, we finally arrive at $\sigma_{(z_1,\dots,z_n)} = \sigma'_{(z_1,\dots,z_n)}$ for all $(z_1,\dots,z_n)\in I_1$, which proves the uniqueness.

So it remains to prove that $\sigma_{(z_1,\dots,z_n)} \perp \sigma_{(z'_1,\dots,z'_n)}$ for every $(z_1,\dots,z_n)$ and $(z'_1,\dots,z'_n)\in I_1$ with $(z_1,\dots,z_n)\neq(z'_1,\dots,z'_n)$. By Theorem 2.4

$$\sum_{(z_1,\dots,z_{n-1},a)\in I_1} \sigma_{(z_1,\dots,z_n)} \perp \sum_{(z_1,\dots,z_{n-1},s)\in I_1} \sigma_{(z_1,\dots,z_n)}$$

and this implies $\sigma_{(z_1,\dots,z_{n-1},a)} \perp \sigma_{(z_1,\dots,z_{n-1},s)}$ for all $z_1,\dots,z_{n-1}$. Also by Theorem 2.4,

$$\sum_{(z_1,\dots,z_{n-2},a,a)\in I_1} \sigma_{(z_1,\dots,z_n)} \perp \sum_{(z_1,\dots,z_{n-2},s,a)\in I_1} \sigma_{(z_1,\dots,z_n)}$$

so $\sigma_{(z_1,\dots,z_{n-2},a,a)} \perp \sigma_{(z_1,\dots,z_{n-2},s,a)}$ for all $z_1,\dots,z_{n-2}$. This is also an argument which can be repeated over and over again. This completes the proof. □

Remember $S^{(i)}_{(z_1,\dots,z_n)} := \operatorname{supp}(\sigma^{(i)}_{(z_1,\dots,z_n)})$ and $S_i := S^{(i)}_{(\tilde{z}_1,\dots,\tilde{z}_n)}$, where $\tilde{z}_j = s$ for $j\neq i$. Since for fixed i we have $\sigma^{(i)}_{(z_1,\dots,z_n)} \perp \sigma^{(i)}_{(z'_1,\dots,z'_n)}$ if $(z_1,\dots,z_n)\neq(z'_1,\dots,z'_n)$, we can assume that $S^{(i)}_{(z_1,\dots,z_n)} \cap S^{(i)}_{(z'_1,\dots,z'_n)} = \emptyset$ if $(z_1,\dots,z_n)\neq(z'_1,\dots,z'_n)$. In Lemma 2.6, we show that the sets $S^{(i)}_{(z_1,\dots,z_n)}$ can be taken in a specific manner. Note that these sets are uniquely defined modulo a set of measure 0. We make a particular choice for the sets $S^{(i)}_{(z_1,\dots,z_n)}$ to be used in the main part of the paper.

LEMMA 2.6. *The sets $S^{(i)}_{(z_1,\dots,z_n)}$ can be chosen such that the following holds:*

(i) *$S_i \cap S_j = \emptyset$ if $i\neq j$.*

(ii) *$S^{(i)}_{(z_1,\dots,z_n)} \cap \operatorname{supp}(\mu_j) = \emptyset$ if $z_j = s$ and $i\neq j$.*

(iii) *Let $i\neq j$ and let $(z_1,\dots,z_n)\in I_i$ and $(z'_1,\dots,z'_n)\in I_j$. Suppose that $z_i = o$, $z_j = a$ and $z'_i = a$, $z'_j = o$ and $z'_l = z_l$ for $l\neq i$ and $l\neq j$. Then $S^{(i)}_{(z_1,\dots,z_n)} = S^{(j)}_{(z'_1,\dots,z'_n)}$.*

PROOF. Notice that (ii) implies (i). We will obtain (ii) and (iii) by making a specific choice for $S^{(i)}_{(z_1,\dots,z_n)}$.

Start by making an arbitrary choice for $\operatorname{supp}(\mu_1),\dots,\operatorname{supp}(\mu_n)$. Look at μ_1 and let $j\neq 1$. Define

$$\sigma_s^{(j)} := \sum_{(z_1,\dots,z_n)\in I_j;\ z_1 = s} \sigma^{(j)}_{(z_1,\dots,z_n)}.$$

Then $\sigma_s^{(j)} \perp \mu_1$, so there exists a set $S_s^{(j)}$ such that $\sigma_s^{(j)}$ is concentrated on $S_s^{(j)}$ and $\mu_1(S_s^{(j)}) = 0$. But in that case it is possible to replace $\operatorname{supp}(\mu_1)$ by $\operatorname{supp}(\mu_1)\backslash S_s^{(j)}$. The same argument can be used for $\mu_2,\dots,\mu_n$, to obtain new supports for $\mu_1,\dots,\mu_n$. Fix these supports and denote them again by $\operatorname{supp}(\mu_j)$.

Define

$$S^{(i)}_{(z_1,\dots,z_n)} = (\operatorname{supp}(\mu_i) \cap \bigcap_{\{j:\ z_j = a\}} \operatorname{supp}(\mu_j))\backslash(\bigcup_{\{j:\ z_j = s\}} \operatorname{supp}(\mu_j)).$$

From this definition, it follows immediately that (ii) and (iii) are satisfied. It remains to show that $\sigma^{(i)}_{(z_1,\dots,z_n)}$ is concentrated on $S^{(i)}_{(z_1,\dots,z_n)}$.

It is obvious that $\sigma^{(i)}_{(z_1,\dots,z_n)}$ is concentrated on (a subset of) $\mathrm{supp}(\mu_i)$. Let j be such that $z_j=s$. From the above construction of the support of μ_j, we may conclude that in that case

$$\sigma^{(i)}_{(z_1,\dots,z_n)}(\mathrm{supp}(\mu_j))=0$$

and thus $\sigma^{(i)}_{(z_1,\dots,z_n)}$ is concentrated on a subset of

$$E:=\mathrm{supp}(\mu_i)\setminus\Big(\bigcup_{j:\, z_j=s}\mathrm{supp}(\mu_j)\Big).$$

Now let j be such that $z_j=a$. Consider $E\setminus\mathrm{supp}(\mu_j)$. On this set we have

$$\mu_j(E\setminus\mathrm{supp}(\mu_j))=0.$$

Since $z_j=a$, it follows that $\sigma^{(i)}_{(z_1,\dots,z_n)}\ll\mu_j$ and thus

$$\sigma^{(i)}_{(z_1,\dots,z_n)}(E\setminus\mathrm{supp}(\mu_j))=0.$$

So $\sigma^{(i)}_{(z_1,\dots,z_n)}$ is indeed concentrated on

$$\Big(\mathrm{supp}(\mu_i)\cap\bigcap_{\{j:\, z_j=a\}}\mathrm{supp}(\mu_j)\Big)\setminus\Big(\bigcup_{\{j:\, z_j=s\}}\mathrm{supp}(\mu_j)\Big),$$

which completes the proof. $\square$

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